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The world-sheet description of A and B branes revisited

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Abstract: We give a manifest supersymmetric description of A and B branes on Kähler manifolds using a completely local $N = 2$ superspace formulation of the world-sheet non-linear σ-model in the presence of a boundary. In particular, we show that an $N = 2$ superspace description of type A boundaries is possible, at least when the background is Kähler. This leads to an elegant and concrete setting for studying coisotropic A branes. Here, an important role is played by the boundary potential, whose precise physical meaning remains to be fully understood. Duality transformations relating A and B branes in the presence of isometries are studied as well.

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Contents

1. Introduction

Non-linear σ-models in two dimensions with an $N = (2, 2)$ supersymmetry, [1]–[4], play a central role in the description of type II superstrings in the absence of R-R fluxes. The interest in these models was recently rekindled as well in the physics as in the mathematics community. For physicists, these models allow for the study of compactifications in the presence of non-trivial NS-NS fluxes, while for mathematicians the models provide a concrete realization of generalized complex geometries. A full off-shell supersymmetric description clarifies the geometry behind these models. The case without boundaries has been studied for more than two decennia and has recently been fully solved in [5] (building on results in e.g. [6]–[9]). Formulating the model in $N = (2, 2)$ superspace allows one to encode the whole (local) geometry in a single scalar function, the Lagrange density. The Lagrange density is a function of scalar superfields satisfying certain constraints. Only three types of superfields are needed [5, 10]: chiral, twisted chiral and semi-chiral superfields.

However, when dealing with D-branes one needs to confront $N = (2, 2)$ non-linear σ-models with boundaries. The presence of boundaries breaks the $N = (2, 2)$ supersymmetry down to an $N = 2$ supersymmetry. While a lot of attention has been paid to these models [11]–[17], their full description in $N = 2$ superspace has not been given yet.

In the present paper we open this study with the simplest case: A and B branes on Kähler manifolds. While it is not too hard to formulate B branes in $N = 2$ superspace [18], type A branes remained enigmatic up till now. As their boundary conditions appear at first sight to be incompatible with the complex structure associated with the $N = (2, 2)$ bulk supersymmetry, one expects a superspace formulation to be subtle.

An important additional motivation for finding an $N = 2$ world-sheet superspace description of A branes is that it provides a new concrete setting for studying coisotropic branes. Indeed, in [19] it was realized that in addition to the usual type A branes wrapping lagrangian cycles, for consistency with mirror symmetry which exchanges A and B branes, one should also include so-called coisotropic branes. Their properties were already established in [19] and later re-derived from a world-sheet point of view in [17]. So far, the only concrete examples appearing in the literature are maximally coisotropic branes on T^4 [19, 20] and $K3$ [21], and coisotropic branes wrapping 5-cycles on T^6, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ and $T^2 \times K3$ [22].

In the next section we revisit the $N = (1, 1)$ supersymmetric non-linear σ-models in the presence of boundaries thereby clarifying some remaining problems. In section 3 we construct $N = 2$ superspace. We show that changing from A to B boundary conditions amounts to interchanging chiral superfields and twisted chiral ones and vice versa. In section 4 we give a detailed description of type A branes followed by a similar description of type B branes in section 5. When certain isometries are present, chiral superfields can be dualized into twisted chiral superfields and vice versa. In section 6 we study these duality transformations in the presence of boundaries. We end with conclusions and an outlook. The study of general non-linear σ-models with boundaries — involving chiral, twisted chiral and semi-chiral fields simultanously — will appear elsewhere [23].

2. From $N = (1, 1)$ to $N = 1$

A non-linear σ-model (with $N \leq (1, 1)$) on some target manifold $\mathcal{M}$ is characterized by a metric g_{ab} and a closed 3-form T_{abc} (known as the torsion, the Kalb-Ramond 3-form or the NS-NS flux) on $\mathcal{M}$. The action in $N = (1, 1)$ superspace is simply,[1]

$$\mathcal{S} = 2 \int d^2\sigma \, d^2\theta \, D_+ X^a D_- X^b \left(g_{ab} + b_{ab} \right), \tag{2.1}$$

where we used the locally defined 2-form potential b_{ab} for the torsion,

$$T_{abc} = -\frac{3}{2} \, \partial_{[a} b_{bc]}. \tag{2.2}$$

[1]Our conventions are given in appendix A.

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We introduce a boundary[2] at $\sigma = 0$ ($\sigma \geq 0$) and $\theta^+ = \theta^-$. This breaks the invariance under translations in both the σ and the $\theta' \equiv \theta^+ - \theta^-$ direction. Put differently, the presence of a boundary breaks the $N = (1,1)$ supersymmetry to an $N = 1$ supersymmetry. We introduce the derivatives,

$$D \equiv D_+ + D_-, \qquad D' \equiv D_+ - D_-, \tag{2.3}$$

which satisfy,

$$D^2 = D'^2 = -\frac{i}{2}\partial_\tau, \qquad \{D, D'\} = -i\,\partial_\sigma. \tag{2.4}$$

Using this, one verifies that

$$-D\,D' = 2D_+\,D_- + \frac{i}{2}\,\partial_\sigma, \tag{2.5}$$

which allows us to write a manifest $N = 1$ supersymmetric lagrangian,

$$\mathcal{S} = -\int d^2\sigma\, d\theta\, D'\left(D_+X^a D_-X^b\,(g_{ab} + b_{ab})\right), \tag{2.6}$$

which — because of eq. (2.5) — differs from the action in the absence of boundaries, eq. (2.1), by a boundary term [25, 18]. Working out the D' derivative yields the action in $N = 1$ boundary superspace obtained in [18],

$$\begin{aligned}\mathcal{S} = \int d^2\sigma\, d\theta\Big(& i\, g_{ab}\, DX^a\partial_\tau X^b - 2i\, g_{ab}\,\partial_\sigma X^a D'X^b + 2i\, b_{ab}\,\partial_\sigma X^a DX^b \\ & -2\, g_{ab}\, D'X^a \nabla D'X^b + 2\, T_{abc}\, D'X^a DX^b DX^c - \frac{2}{3}\, T_{abc}\, D'X^a D'X^b D'X^c\Big),\end{aligned} \tag{2.7}$$

where,

$$\nabla D'X^a \equiv DD'X^a + \left\{{a \atop bc}\right\} DX^b D'X^c, \tag{2.8}$$

and both X^a and $D'X^a$ should now be viewed as independent $N = 1$ superfields. Note that when $b_{ab} = \partial_a A_b - \partial_b A_a$, we can rewrite eq. (2.7) as,

$$\begin{aligned}\mathcal{S} = & \int d^2\sigma\, d\theta\Big(i\, g_{ab}\, DX^a\partial_\tau X^b - 2i\, g_{ab}\,\partial_\sigma X^a D'X^b - 2\, g_{ab}\, D'X^a \nabla D'X^b\Big) \\ & + 2i \int d\tau d\theta\, A_a\, DX^a.\end{aligned} \tag{2.9}$$

Varying the action eq. (2.6)[3] or eq. (2.7) yields a boundary term,

$$\delta\mathcal{S}\big|_{\text{boundary}} = -2i \int d\tau d\theta\, \delta X^a \left(g_{ab}\, D'X^b - b_{ab}\, DX^b\right). \tag{2.10}$$

[2]As far as we know, the first place where superspaces with boundaries were introduced and used was in [24].

[3]Where one uses that $\int d^2\sigma d\theta D'\, D_\pm = -(i/2) \int d\tau d\theta$.

This boundary term will only vanish if suitable boundary conditions are imposed. In order to do so we introduce a (1,1) tensor $R(X)^a{}_b$ [11, 15, 16, 18] which satisfies,

$$R^a{}_c\, R^c{}_b = \delta^a_b, \tag{2.11}$$

and projection operators $\mathcal{P}_\pm$,

$$\mathcal{P}^a_{\pm b} \equiv \frac{1}{2}\left(\delta^a_b \pm R^a{}_b\right). \tag{2.12}$$

With this we impose Dirichlet boundary conditions,

$$\mathcal{P}^a_{-b}\, \delta X^b = 0. \tag{2.13}$$

Using eq. (2.13), one verifies that the boundary term eq. (2.10) vanishes, provided one imposes Neumann boundary conditions,

$$\mathcal{P}_{+ba}\, D' X^b = \mathcal{P}^b_{+a}\, b_{bc}\, D X^c, \tag{2.14}$$

as well. If in addition we assume — for which at this point, as we will demonstrate in an example later on, there is no necessary reason — that,

$$g_{ac} R^c{}_b = g_{bc} R^c{}_a, \tag{2.15}$$

or $R_{ab} = R_{ba}$, then we can rewrite eq. (2.14) as,

$$\mathcal{P}^a_{+b}\, D' X^b = \mathcal{P}^a_{+c}\, b^c{}_d \mathcal{P}^d_{+b}\, D X^b. \tag{2.16}$$

Invariance of the Dirichlet boundary conditions under what remains of the super-Poincaré transformations implies that on the boundary,

$$\mathcal{P}^a_{-b}\, D X^b = \mathcal{P}^a_{-b}\, \partial_\tau X^b = 0, \tag{2.17}$$

hold as well. Using $D^2 = -i/2\, \partial_\tau$, we get from eq. (2.17) the integrability conditions,[4]

$$0 = \mathcal{P}^d_{+[b} \mathcal{P}^e_{+c]} \mathcal{P}^a_{+d,e} = -\frac{1}{2}\, \mathcal{P}^a_{-e}\, \mathcal{N}^e{}_{bc}[R, R]. \tag{2.18}$$

These conditions guarantee the existence of adapted coordinates $X^{\hat a}$, $\hat a \in \{p+1, \cdots, d\}$, with $p \le d$ the rank of $\mathcal{P}_+$ such that the Dirichlet boundary conditions, eq. (2.13) are simply given by,

$$X^{\hat a} = \text{ constant}, \qquad \forall\, \hat a \in \{p+1, \cdots, d\}. \tag{2.19}$$

Writing the remainder of the coordinates as $X^{\check a}$, $\check a \in \{1, \cdots, p\}$, we get the Neumann boundary conditions, eq. (2.14), in our adapted coordinates,

$$g_{\check a b}\, D' X^b = b_{\check a \check b}\, D X^{\check b}, \tag{2.20}$$

[4]Out of two $(1,1)$ tensors $R^a{}_b$ and $S^a{}_b$, one constructs a $(1,2)$ tensor $\mathcal{N}[R,S]^a{}_{bc}$, the Nijenhuis tensor, as $\mathcal{N}[R,S]^a{}_{bc} = R^a{}_d S^d{}_{[b,c]} + R^d{}_{[b} S^a{}_{c],d} + R \leftrightarrow S$.

where b is summed from 1 to d and where we used that $DX^{\hat{b}}$ vanishes on the boundary. Concluding, the action eq. (2.6) together with the boundary conditions eqs. (2.19) and (2.20), describe open strings in the presence of a Dp-brane whose position is determined by eq. (2.19).

Let us end this section with an example. We start with a very simple configuration consisting of a D2-brane on a 2-torus with coordinates X^1 and X^2 in the presence of a U(1) magnetic background[5] $F_{12} = F(X^1)$ (note that $\partial_2 F = 0$). The action is,

$$\mathcal{S}_{D2} = -\int d^2\sigma d\theta D' \Big(D_+X^1D_-X^1 + D_+X^2D_-X^2 + F(X^1)\left(D_+X^1D_-X^2 - D_+X^2D_-X^1\right)\Big), \tag{2.21}$$

and the boundary conditions are Neumann in all directions,

$$D'X^1 = +\,F(X^1)\,DX^2, \qquad D'X^2 = -\,F(X^1)\,DX^1. \tag{2.22}$$

Making a T-duality transformation along the X^2 direction[6] yields a D1-brane with action,

$$\mathcal{S}_{D1} = -\int d^2\sigma d\theta D' \Big((1+F^2)\,D_+X^1D_-X^1 + D_+X^2D_-X^2 + F\left(D_+X^1D_-X^2 + D_+X^2D_-X^1\right)\Big), \tag{2.23}$$

and boundary conditions,

$$\begin{aligned} \delta X^2 &= 0, \\ (1+F^2)D'X^1 + F\,D'X^2 &= 0. \end{aligned} \tag{2.24}$$

Comparing eq. (2.23) to eq. (2.6), we read off the (flat) metric: $g_{11} = 1+F^2$, $g_{12} = F$ and $g_{22} = 1$. Comparing the boundary conditions eq. (2.24) with eqs. (2.13) and (2.14), we get $R^1{}_1 = 1$, $R^1{}_2 = 2F/(1+F^2)$, $R^2{}_1 = 0$ and $R^2{}_2 = -1$. One verifies that for this choice of $R^a{}_b$, $R_{ab} = R_{ba}$ holds. Note that we might as well have chosen $R^1{}_1 = -R^2{}_2 = 1$ and $R^1{}_2 = R^2{}_1 = 0$ which also reproduce the boundary conditions eq. (2.24). However for this choice we have $R_{ab} \neq R_{ba}$.

The D1-brane configuration described here is fairly standard. Indeed, take the Dirichlet boundary condition to be $X^2 = 0$ and change coordinates,

$$Y^1 = X^1, \qquad Y^2 = X^2 + A(X^1), \tag{2.25}$$

where the potential $A(X^1)$ is defined by $F(X^1) = \partial_1 A(X^1)$. In these coordinates the metric becomes the standard one, $g_{ab} = \delta_{ab}$, and the D1-brane is defined by $Y^2 = A(Y^1)$, where Y^1 assumes the role of worldvolume coordinate. Taking a constant magnetic field, $F = \tan\theta$, we recognize the system as a straight D1-brane rotated in the Y^1Y^2-plane over an angle θ with respect to the Y^1-axis.

[5]Whenever b_{ab} is closed, we will denote it by F_{ab}.

[6]A simple way to do this is by gauging the isometry $X^2 \to X^2 +$ constant and — using Lagrange multipliers — imposing that the gauge fields are pure gauge. Integrating over the gauge fields yields the T-dual model, see e.g. [26].

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3. N=2 superspace

3.1 $N = (2,2)$ supersymmetry in the absence of boundaries

Even without boundaries, promoting the $N = (1,1)$ supersymmetry of the action in eq. (2.1) to an $N = (2,2)$ is a non-trivial operation which introduces a lot of additional geometric structure in the model. The most general extra supersymmetry transformations — consistent with dimensions and Poincaré symmetry — are of the form,

$$\delta X^a = \varepsilon^+ J^a_{+b}(X)\, D_+ X^b + \varepsilon^- J^a_{-b}(X)\, D_- X^b, \tag{3.1}$$

which requires the introduction of two (1,1) tensors J_+ and J_-. Requiring the supersymmetry algebra to close *on-shell*, one finds that both J_+ and J_- must be complex structures,

$$\begin{aligned} J^a_{\pm c}\, J^c_{\pm b} &= -\delta^a_b, \\ N[J_\pm, J_\pm]^a{}_{bc} &= 0. \end{aligned} \tag{3.2}$$

Apart from requiring that the $N = (2,2)$ supersymmetry algebra is satisfied, we have to demand that the action eq. (2.1) is invariant under the transformations eq. (3.1). This yields additional conditions. The metric has to be hermitian with respect to *both* complex structures,[7]

$$J^c_{\pm a}\, J^d_{\pm b}\, g_{cd} = g_{ab}\,. \tag{3.3}$$

Furthermore, both complex structures have to be covariantly constant,

$$0 = \nabla^\pm_c\, J^a_{\pm b} \equiv \partial_c\, J^a_{\pm b} + \Gamma^a_{\pm dc} J^d_{\pm b} - \Gamma^d_{\pm bc} J^a_{\pm d}\,, \tag{3.4}$$

with the connections $\Gamma_\pm$ given by,

$$\Gamma^a_{\pm bc} \equiv \left\{{a \atop bc}\right\} \pm T^a{}_{bc}\,. \tag{3.5}$$

A complex manifold with the above additional properties is called bihermitian. When the torsion vanishes, this type of geometry reduces to the usual Kähler geometry.

When calculating the algebra explicitly one finds that the terms in the algebra which do not close off-shell are proportional to the commutator of the complex structures $[J_+, J_-]$. In order to obtain an off-shell closing formulation of the model, one expects that $\ker\,[J_+, J_-]$ can be described without any additional auxiliary fields while the description of $\mathrm{coker}[J_+, J_-]$ will require the introduction of new auxiliary fields. This picture was already suggested in [8] and [9] (see also [7]) and was shown in [5] to be correct. Roughly speaking one gets that when writing $\ker\,[J_+, J_-] = \ker(J_+ - J_-) \oplus \ker(J_+ + J_-)$, $\ker(J_+ - J_-)$ and $\ker(J_+ + J_-)$ resp. can be integrated to chiral and twisted chiral multiplets resp. [2]. Semi-chiral multiplets [6] are required for the description of $\mathrm{coker}[J_+, J_-]$.

[7] This implies the existence of two two-forms $\omega^\pm_{ab} = -\omega^\pm_{ba} = g_{ac} J^c_{\pm b}$. In general they are not closed. Using eq. (3.4), one shows that $\omega^\pm_{[ab,c]} = \mp 2 J^d_{\pm[a} T_{bc]d} = \mp(2/3) J^d_{\pm a} J^e_{\pm b} J^f_{\pm c} T_{\mathrm{def}}$, where for the last step we used the fact that the Nijenhuis tensors vanish.

In the present paper we will focus on chiral and twisted chiral multiplets, i.e. we assume that J_+ and J_- commute.[8] These fields in $N = (2,2)$ superspace (once more we refer to the appendix for conventions) satisfy the constraints $\hat{D}_\pm X^a = J^a_{\pm b} D_\pm X^b$ where J_+ and J_- can be simultaneously diagonalized. When the eigenvalues of J_+ and J_- have the same (the opposite) sign we have chiral (twisted chiral) superfields. Explicitly, we get that chiral superfields X^α, $\alpha \in \{1, \cdots, m\}$, satisfy,

$$\hat{D}_\pm X^\alpha = +i\, D_\pm X^\alpha, \qquad \hat{D}_\pm X^{\bar\alpha} = -i\, D_\pm X^{\bar\alpha}. \tag{3.6}$$

Twisted chiral superfields X^μ, $\mu \in \{1, \cdots, n\}$ satisfy,

$$\hat{D}_\pm X^\mu = \pm i\, D_\pm X^\mu, \qquad \hat{D}_\pm X^{\bar\mu} = \mp i\, D_\pm X^{\bar\mu}. \tag{3.7}$$

The most general action involving these superfields is given by,

$$\mathcal{S} = \int d^2\sigma\, d^2\theta\, d^2\hat\theta\, V(X, \bar X), \tag{3.8}$$

where the Lagrange density $V(X, \bar X)$ is an arbitrary real function of the chiral and twisted chiral superfields. Passing to $N = (1,1)$ superspace and comparing the result to eq. (2.1), allows one to identify the metric and the torsion potential,[9]

$$\begin{aligned} g_{\alpha\bar\beta} &= +V_{\alpha\bar\beta}, \qquad & g_{\mu\bar\nu} &= -V_{\mu\bar\nu}, \\ b_{\alpha\bar\nu} &= -V_{\alpha\bar\nu}, \qquad & b_{\mu\bar\beta} &= +V_{\mu\bar\beta}, \end{aligned} \tag{3.9}$$

where all other components of g and b vanish. When writing $V_{\alpha\bar\beta}$, we mean $\partial_\alpha \partial_{\bar\beta} V$ etc. Note that when only one type of superfield is present, the target manifold is Kähler, which is the case in which we are presently interested. The case where both of them are simultaneously present will be discussed elsewhere [23].

3.2 From $N = (2,2)$ to $N = 2$

We now assume that in the bulk — far away from the boundary — the model exhibits an $N = (2,2)$ supersymmetry as described in the previous subsection. We expect the boundary to break half of the supersymmetries, so we will go from $N = (2,2)$ to $N = 2$. In order to handle this we rewrite eq. (3.1) as,

$$\delta X^a = \varepsilon\, J^{(+)a}{}_b\, DX^b + \varepsilon\, J^{(-)a}{}_b\, D'X^b + \varepsilon'\, J^{(-)a}{}_b\, DX^b + \varepsilon'\, J^{(+)a}{}_b\, D'X^b, \tag{3.10}$$

where,

$$\begin{aligned} \varepsilon &\equiv \frac{1}{2}(\varepsilon^+ + \varepsilon^-), \qquad \varepsilon' \equiv \frac{1}{2}(\varepsilon^+ - \varepsilon^-), \\ J^{(\pm)} &\equiv \frac{1}{2}\,(J_+ \pm J_-)\,. \end{aligned} \tag{3.11}$$

[8] As already mentioned in the introduction we relegate the study of the most general case — which includes the semi-chiral superfields — to a forthcoming paper [23].

[9] Indices from the beginning of the Greek alphabet, α, β, γ, ... denote chiral fields while indices from the middle of the alphabet, μ, ν, ρ, ... denote twisted chiral fields.

Whenever the ε supersymmetry is preserved, one talks about B-type boundary conditions, while preservation of the ε' supersymmetry corresponds to what are called A-type boundary conditions. One sees that switching from B-type to A-type amounts to replacing ε by ε' and $J^{(\pm)}$ by $J^{(\mp)}$. In $N=(2,2)$ superspace, B-boundary conditions correspond to a boundary $\theta' \equiv (\theta^+ - \theta^-)/2 = 0$ and $\hat\theta' \equiv (\hat\theta^+ - \hat\theta^-)/2 = 0$. A-type boundary conditions on the other hand correspond to $\theta' \equiv (\theta^+ - \theta^-)/2 = 0$ and $\hat\theta' \equiv (\hat\theta^+ + \hat\theta^-)/2 = 0$. For B-type boundaries we define,

$$
\begin{aligned}
D &\equiv D_+ + D_-, \qquad \hat D \equiv \hat D_+ + \hat D_-,\\
D' &\equiv D_+ - D_-, \qquad \hat D' \equiv \hat D_+ - \hat D_-,
\end{aligned}
\tag{3.12}
$$

where unaccented derivatives refer to translations in the invariant directions. When dealing with A-type boundaries, the role of $\hat D$ and $\hat D'$ are interchanged. For the moment we will focus on B-type boundaries. Later on we will see that this does not present any restriction as switching from one type of boundary conditions to another will just amount to interchanging chiral for twisted chiral superfields and vice-versa. The derivatives defined in eq. (3.12) satisfy,

$$
\begin{aligned}
D^2 = \hat D^2 = D'^2 = \hat D'^2 &= -\frac{i}{2}\partial_\tau,\\
\{D, D'\} = \{\hat D, \hat D'\} &= -i\partial_\sigma,
\end{aligned}
\tag{3.13}
$$

and all other anti-commutators vanish.

Let us now turn to the superfields. In the bulk we had chiral, twisted chiral and semi-chiral superfields. In the present paper we focus on chiral and twisted chiral superfields. From eqs. (3.6) and (3.12) we get for the chiral fields,

$$
\begin{aligned}
\hat D X^\alpha &= +iDX^\alpha, \quad \hat D X^{\bar\alpha} = -iDX^{\bar\alpha}\\
\hat D' X^\alpha &= +iD'X^\alpha, \quad \hat D' X^{\bar\alpha} = -iD'X^{\bar\alpha},
\end{aligned}
\tag{3.14}
$$

where $\alpha,\ \bar\alpha \in \{1, \cdots, m\}$. Passing from $N=(2,2)$ — parametrized by the Grassmann coordinates θ, $\hat\theta$, θ' and $\hat\theta'$ — to $N=2$ superspace – parametrized by θ and $\hat\theta$ — we get X^α, $X^{\bar\alpha}$, $D'X^\alpha$ and $D'X^{\bar\alpha}$ as $N=2$ superfields and they satisfy the constraints,

$$
\begin{aligned}
\hat D X^\alpha &= +i\, DX^\alpha, & \hat D X^{\bar\alpha} &= -i\, DX^{\bar\alpha},\\
\hat D\, D' X^\alpha &= +i\, D\, D'X^\alpha - \partial_\sigma X^\alpha, & \hat D\, D'X^{\bar\alpha} &= -i\, D\, D'X^{\bar\alpha} + \partial_\sigma X^{\bar\alpha}.
\end{aligned}
\tag{3.15}
$$

For twisted chiral superfields we get instead, when combining eqs. (3.7) and (3.12),

$$
\begin{aligned}
\hat D X^\mu &= +iD'X^\mu, \quad \hat D X^{\bar\mu} = -iD'X^{\bar\mu},\\
\hat D' X^\mu &= +iDX^\mu, \quad \hat D' X^{\bar\mu} = -iDX^{\bar\mu},
\end{aligned}
\tag{3.16}
$$

with $\mu,\ \bar\mu \in \{1, \cdots, n\}$. Passing again from $N=(2,2)$ to $N=2$ superspace, we now get X^μ, $X^{\bar\mu}$, $D'X^\mu$ and $D'X^{\bar\mu}$ as $N=2$ superfields satisfying the constraints,

$$
\begin{aligned}
\hat D X^\mu &= +i\, D'X^\mu, \quad \hat D X^{\bar\mu} = -i\, D'X^{\bar\mu},\\
\hat D\, D'X^\mu &= -\frac{1}{2}\dot X^\mu, \quad \hat D\, D'X^{\bar\mu} = +\frac{1}{2}\dot X^{\bar\mu}.
\end{aligned}
\tag{3.17}
$$

So in $N = 2$ superspace, the twisted chiral superfields X^μ and $X^{\bar\mu}$ are unconstrained superfields.

It is important to note that, had we used A-type boundaries instead of B-type, we would have gotten exactly the same expressions but with the roles of chiral and twisted chiral fields interchanged. We will return to duality transformations interchanging chiral for twisted chiral fields and vice versa in section 6.

Once more one immediately verifies that the difference between the fermionic measure $D_+D_-\hat{D}_+\hat{D}_-$ and $D\hat{D}D'\hat{D}'$ is just a boundary term. So the (al)most general $N = 2$ invariant action which reduces to the usual action far away from the boundary we can write down is,

$$\mathcal{S} = \int d^2\sigma\, d\theta d\hat{\theta}\, D'\hat{D}'\, V(X, \bar{X}), \tag{3.18}$$

where $V(X, \bar{X})$ is an arbitrary function of the (bulk) superfields. In fact, when boundaries are present, we can still generalize the previous by adding a boundary term,

$$\mathcal{S} = \int d^2\sigma\, d\theta d\hat{\theta}\, D'\hat{D}'\, V(X, \bar{X}) + i \int d\tau\, d\theta d\hat{\theta}\, W(X, \bar{X}), \tag{3.19}$$

with $W(X, \bar{X})$ an arbitrary function of the (bulk) superfields.

4. Type A branes

Type A branes on Kähler manifolds are described in terms of twisted chiral fields, eqs. (3.16) and (3.17). The most general $N = 2$ supersymmetric action we can write down is,

$$\mathcal{S} = \int d^2\sigma d^2\theta D'\hat{D}'\, V(X, \bar{X}) + i \int d\tau d^2\theta\, W(X, \bar{X}). \tag{4.1}$$

with $V(X, \bar{X})$ and $W(X, \bar{X})$ arbitrary functions of the twisted chiral fields. Working out the $\hat{D}'$ and D' derivatives using the constraints eq. (3.16) gives the action in $N = 2$ boundary superspace,

$$\begin{aligned}\mathcal{S} \,=\, &\int d^2\sigma d^2\theta\, \left(2iV_{\bar\mu\nu}\, D'X^{\bar\mu} DX^\nu - 2iV_{\mu\bar\nu}\, D'X^\mu DX^{\bar\nu} + V_\mu\, \partial_\sigma X^\mu - V_{\bar\mu}\, \partial_\sigma X^{\bar\mu}\right)\\ &+ i \int d\tau d^2\theta\, W.\end{aligned} \tag{4.2}$$

It is quite interesting to note that even here — contrary to what is sometimes claimed — the theory remains invariant under Kähler transformations. Indeed, one readily verifies that,

$$\begin{aligned} V(X, \bar{X}) &\to V'(X, \bar{X}) = V(X, \bar{X}) + f(X) + \bar{f}(\bar{X}),\\ W(X, \bar{X}) &\to W'(X, \bar{X}) = W(X, \bar{X}) + i\left(f(X) - \bar{f}(\bar{X})\right), \end{aligned} \tag{4.3}$$

leaves the action eq. (4.2) invariant. Performing the integral over $\hat{\theta}$ in eq. (4.2) yields eq. (2.7) with vanishing torsion, $T = 0$, and a Kähler metric given by $g_{\mu\bar\nu} = V_{\mu\bar\nu}$. However, we find that eq. (2.7) comes with an extra, non-standard boundary term of the form,[10]

$$\mathcal{S}_{\text{extra}} = i \int d\tau d\theta \Big((V + i\,W)_{\mu}\, D'X^{\mu} + (V - i\,W)_{\bar\mu}\, D'X^{\bar\mu} \Big) . \tag{4.4}$$

Varying the action in eq. (4.2),[11] yields besides the standard bulk equations of motion a boundary contribution given by,

$$\delta\mathcal{S}\Big|_{\text{boundary}} = \int d\tau d^2\theta \Big((V + i\,W)_{\mu}\, \delta X^{\mu} - (V - i\,W)_{\bar\mu}\, \delta X^{\bar\mu} \Big) . \tag{4.5}$$

Both eqs. (4.4) and (4.5) indicate that the choice of boundary conditions will be subtle here.

In order to get a feeling of what is going on, we first look at the simplest situation where there is only a single twisted chiral field (which we call w), i.e. $n = 1$. The model is characterized by two potentials $V(w, \bar w)$ and $W(w, \bar w)$. We get that the boundary term in the variation of the action, eq. (4.5), vanishes provided we impose the Dirichlet boundary condition,

$$\delta w = R^{w}{}_{\bar w}\, \delta\bar w, \tag{4.6}$$

with,

$$R^{w}{}_{\bar w} \equiv \frac{V_{\bar w} - i\,W_{\bar w}}{V_{w} + i\,W_{w}}. \tag{4.7}$$

Eq. (4.6) implies,

$$\hat D w = R^{w}{}_{\bar w}\, \hat D\bar w, \tag{4.8}$$

which using the constraints eq. (3.17) reduces to the Neumann boundary condition,

$$D'w + R^{w}{}_{\bar w}\, D'\bar w = 0. \tag{4.9}$$

So the σ-model describes open strings propagating on a Kähler manifold with Kähler potential V in the presence of a D1-brane wrapped on a lagrangian submanifold (a trivial notion in two dimensions) whose position is determined by eq. (4.6). In order to make contact with the example discussed at the end of section 2, we restrict ourselves to flat space, i.e. $V = (w + \bar w)^2/2$, and assume that W has the form $W = W(w + \bar w)$. Using the coordinates defined in eq. (2.25), we identify $w = (Y^1 + i\,Y^2)/\sqrt{2}$ and we find,

$$W = -(w + \bar w)\, Q'(w + \bar w) + Q(w + \bar w), \tag{4.10}$$

where a prime denotes a derivative with respect to either w or $\bar w$ and $Q(w + \bar w)$ is a "prepotential" for F which appears in eqs. (2.23) and (2.24),

$$F = \partial_w \partial_{\bar w}\, Q(w + \bar w). \tag{4.11}$$

[10]This unusual boundary term was already noticed in [18]. In order to recover the standard boundary term — as in eq. (2.9), we will need non-trivial Neumann boundary conditions eq. (2.14).

[11]When varying we use the fact that the $N = 2$ superfields X^{μ} and $X^{\bar\mu}$ are unconstraind, while $\delta D'X^{\mu} = -i\hat D\delta X^{\mu}$ and similarly for $\delta D'X^{\bar\mu}$.

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We get here,

$$R^{w}{}_{\bar{w}} = \frac{1 + i\,Q''}{1 - i\,Q''} = \frac{1 + i\,F}{1 - i\,F}, \tag{4.12}$$

and this corresponds to the first choice (i.e. the one for which $R_{ab} = R_{ba}$ holds) for $R^a{}_b$ made in section 2. Using this we obtain the boundary conditions,

$$-i(w - \bar{w}) - Q'(w + \bar{w}) = \text{ constant}, \tag{4.13}$$

and,

$$D'w + D'\bar{w} = i\,Q''(w + \bar{w})\big(D'w - D'\bar{w}\big). \tag{4.14}$$

The resulting model is precisely the one discussed in section 2, however now in a manifest $N = 2$ supersymmetric setting. The extended supersymmetry fixed the choice of $R^a{}_b$. Note that it is the potential W which allows us to tune the precise location of the D1-brane.

We now turn to the general case. The Dirichlet boundary conditions can be written as,

$$\delta X^\mu = R^\mu{}_{\bar{\nu}}\,\delta X^{\bar{\nu}} + R^\mu{}_\nu\,\delta X^\nu. \tag{4.15}$$

Invariance of the boundary conditions under the supersymmetry transformations implies,

$$\hat{D} X^\mu = R^\mu{}_{\bar{\nu}}\,\hat{D} X^{\bar{\nu}} + R^\mu{}_\nu\,\hat{D} X^\nu, \tag{4.16}$$

which using the constraints eq. (3.16) results in,

$$\big(\mathcal{P}_+ D'X\big)^\mu = R^\mu{}_\nu\,D'X^\nu. \tag{4.17}$$

Requiring this to be compatible with $\mathcal{P}_+\mathcal{P}_+ = \mathcal{P}_+$ yields,

$$\begin{aligned} R^\mu{}_\rho R^\rho{}_\nu &= R^\mu{}_\nu, \\ R^\mu{}_{\bar{\rho}} R^{\bar{\rho}}{}_{\bar{\nu}} &= 0. \end{aligned} \tag{4.18}$$

Combining this with $R^a{}_c R^c{}_b = \delta^a_b$ gives in addition,

$$\begin{aligned} R^\mu{}_{\bar{\rho}} R^{\bar{\rho}}{}_\nu &= \delta^\mu_\nu - R^\mu{}_\nu, \\ R^\mu{}_\rho R^\rho{}_{\bar{\nu}} &= 0, \end{aligned} \tag{4.19}$$

as well. Decomposing the complexified tangent space $T_\mathcal{M}$ as $T_\mathcal{M} = T^{(1,0)}_\mathcal{M} \oplus T^{(0,1)}_\mathcal{M}$, we see that eqs. (4.18) and (4.19) imply the existence of projection operators $\pi_\pm : T^{(1,0)}_\mathcal{M} \to T^{(1,0)}_\mathcal{M}$,

$$\begin{aligned} \pi^\mu_{+\nu}\,\delta X^\nu &\equiv R^\mu{}_\nu\,\delta X^\nu, \\ \pi^\mu_{-\nu}\,\delta X^\nu &\equiv R^\mu{}_{\bar{\rho}} R^{\bar{\rho}}{}_\nu\,\delta X^\nu. \end{aligned} \tag{4.20}$$

This allows us to rewrite the Dirichlet boundary conditions as,

$$(\pi_-\delta X)^\mu = R^\mu{}_{\bar{\nu}}\,\delta X^{\bar{\nu}}. \tag{4.21}$$

For the Neumann directions we get,

$$\begin{aligned}\left(\pi_+\mathcal{P}_+D'X\right)^\mu &= R^\mu{}_\nu D'X^\nu,\\ \left(\pi_-\mathcal{P}_+D'X\right)^\mu &= 0.\end{aligned}\tag{4.22}$$

Comparing this to eq. (2.14) we conclude that we will have a non-degenerate magnetic background in the π_+ directions while the magnetic field vanishes in the π_- directions.

We first consider the case for which $\ker\pi_- = \emptyset$, i.e. the only non-vanishing components of R are $R^\mu{}_{\bar\nu}$ and $R^{\bar\mu}{}_\nu$. The Dirichlet boundary conditions are simply,

$$\delta X^\mu = R^\mu{}_{\bar\nu}\,\delta X^{\bar\nu}.\tag{4.23}$$

This implies the integrability conditions,

$$R^\mu{}_{[\bar\nu,\bar\rho]} = R^\sigma{}_{[\bar\nu}R^\mu{}_{\bar\rho],\sigma}.\tag{4.24}$$

The boundary term in the variation of the action, eq. (4.5), will vanish provided,

$$\left(V+i\,W\right)_\mu\delta X^\mu = \left(V-i\,W\right)_{\bar\mu}\delta X^{\bar\mu},\tag{4.25}$$

which implies that,

$$\left(V+i\,W\right)_\mu R^\mu{}_{\bar\nu} = \left(V-i\,W\right)_{\bar\nu},\tag{4.26}$$

should hold. As a consequence, we find that besides eq. (4.25), $\left(V+i\,W\right)_\mu DX^\mu = \left(V-i\,W\right)_{\bar\mu}DX^{\bar\mu}$ and $\left(V+i\,W\right)_\mu \dot X^\mu = \left(V-i\,W\right)_{\bar\mu}\dot X^{\bar\mu}$ hold as well. Using $D^2=-(i/2)\partial_\tau$ we get that the previous is consistent provided,

$$V_{\mu\bar\rho}\,R^{\bar\rho}{}_\nu = V_{\nu\bar\rho}\,R^{\bar\rho}{}_\mu,\tag{4.27}$$

holds, i.e. $R_{\mu\nu}=R_{\nu\mu}$. Introducing a set of real worldvolume coordinates σ^τ, $\tau\in\{1,\cdots n\}$, we get that eq. (4.24) guarantees that,

$$\frac{\partial X^\mu}{\partial\sigma^\tau} = R^\mu{}_{\bar\nu}\frac{\partial X^{\bar\nu}}{\partial\sigma^\tau},\tag{4.28}$$

is satisfied. With this and eq. (4.27), one finds immediately that the pullback of the Kähler two-form to the worldvolume of the brane vanishes. This shows that, whenever $\ker\pi_-=\emptyset$, we have a Dn-brane which wraps an isotropic submanifold of maximal dimension, i.e. a lagrangian submanifold.[12] From eq. (4.23), we get that $\hat DX^\mu = R^\mu{}_{\bar\nu}\,\hat DX^{\bar\nu}$, which using the constraints gives the Neumann boundary conditions,

$$D'X^\mu + R^\mu{}_{\bar\nu}\,D'X^{\bar\nu} = 0,\tag{4.29}$$

from which it follows that for a lagrangian D-brane the magnetic field is necessarily zero. In other words, a lagrangian D-brane can only carry a line bundle with flat connection.

[12]For the definition of isotropic and lagrangian submanifolds, see appendix B.

We now come to the case where $\ker \pi_- \neq \emptyset$. In order to proceed, we assume the existence of adapted coordinates $X^{\check\mu}$ and $X^{\hat\mu}$ (and their complex conjugates), $\check\mu, \check\nu, \cdots \in \{1, \cdots, k\}$ and $\hat\mu, \hat\nu, \cdots \in \{k+1, \cdots, n\}$, such that the only non-vanishing components of π_+ and π_- are $\pi_-{}^{\hat\mu}{}_{\hat\nu} = \delta^{\hat\mu}_{\hat\nu}$ and $\pi_+{}^{\check\mu}{}_{\check\nu} = \delta^{\check\mu}_{\check\nu}$. The only non-vanishing components of R are then $R^{\hat\mu}{}_{\bar{\hat\nu}}(\hat X, \check X)$ and $R^{\check\mu}{}_{\check\nu} = \delta^{\check\mu}_{\check\nu}$. The Dirichlet boundary conditions become,

$$\delta X^{\hat\mu} = R^{\hat\mu}{}_{\bar{\hat\nu}} \delta X^{\bar{\hat\nu}}. \tag{4.30}$$

The resulting integrability conditions imply that $R^{\hat\mu}{}_{\bar{\hat\nu}}$ does not depend on $X^{\check\mu}$ or $X^{\bar{\check\mu}}$ (so $R^{\hat\mu}{}_{\bar{\hat\nu}} = R^{\hat\mu}{}_{\bar{\hat\nu}}(\hat X)$) and,

$$R^{\hat\mu}{}_{[\bar{\hat\nu},\bar{\hat\rho}]} = R^{\hat\sigma}{}_{[\bar{\hat\nu}} R^{\hat\mu}{}_{\bar{\hat\rho}],\hat\sigma}. \tag{4.31}$$

A necessary — but not sufficient — condition for the vanishing of the boundary term in eq. (4.5) is that,

$$(V + i\,W)_{\hat\mu}\, \delta X^{\hat\mu} = (V - i\,W)_{\bar{\hat\mu}}\, \delta X^{\bar{\hat\mu}}, \tag{4.32}$$

which requires that,

$$(V + i\,W)_{\hat\mu}\, R^{\hat\mu}{}_{\bar{\hat\nu}} = (V - i\,W)_{\bar{\hat\nu}}\,, \tag{4.33}$$

should hold. Eq. (4.32) also implies that,

$$V_{\hat\mu\bar{\hat\rho}} R^{\bar{\hat\rho}}{}_{\hat\nu} = V_{\hat\nu\bar{\hat\rho}} R^{\bar{\hat\rho}}{}_{\hat\mu}, \tag{4.34}$$

or $R_{\hat\mu\hat\nu} = R_{\hat\nu\hat\mu}$. From eq. (4.30) and the bulk constraints eq. (3.16) we obtain part of the Neumann boundary conditions,

$$D'\, X^{\hat\mu} + R^{\hat\mu}{}_{\bar{\hat\nu}} D'\, X^{\bar{\hat\nu}} = 0. \tag{4.35}$$

With this the boundary term in the variation of the action, eq. (4.5) does not vanish yet. Denoting the coordinates $X^{\check\mu}$ and $X^{\bar{\check\mu}}$ collectively by $X^{\check a}$ and introducing the canonical complex structure $J^{\check a}{}_{\check b}$,[13] we rewrite eq. (4.5) using eq. (4.32),

$$\delta\mathcal{S}\Big|_{\text{boundary}} = -i \int d\tau d^2\theta \left(V_{\check b} J^{\check b}{}_{\check a} - W_{\check a} \right) \delta X^{\check a}. \tag{4.36}$$

Eq. (4.22) suggested the presence of a non-degenerate magnetic field $F_{\check a\check b}$ which implies Neumann boundary conditions of the form,

$$D' X^{\check a} = F^{\check a}{}_b\, D X^b, \tag{4.37}$$

where indices without checks or hats run from 1 through $d = 2n$. Using the fact that the bulk constraints eq. (3.16) can be rewritten as,

$$\hat D X^{\check a} = J^{\check a}{}_{\check b}\, D' X^{\check b}, \tag{4.38}$$

[13] Its nonvanishing components are $J^{\check\mu}{}_{\check\nu} = +i\,\delta^{\check\mu}_{\check\nu}$ and $J^{\bar{\check\mu}}{}_{\bar{\check\nu}} = -i\,\delta^{\bar{\check\mu}}_{\bar{\check\nu}}$.

we propose Neumann boundary conditions of the form,

$$\hat{D}X^{\check{a}} = K^{\check{a}}{}_b\, DX^b, \tag{4.39}$$

with $K^{\check{a}}{}_b = J^{\check{a}}{}_{\check{c}} F^{\check{c}}{}_b$. Combining $\hat{D}^2 = -(i/2)\partial_\tau$ with eq. (4.39) we get that eq. (4.39) must be of the form,

$$\hat{D}X^{\check{a}} = K^{\check{a}}{}_{\check{b}} DX^{\check{b}}, \tag{4.40}$$

and $K^{\check{a}}{}_{\check{b}}$ is a complex structure (i.e. it squares to -1 and its Nijenhuis tensor vanishes) which depends only on $X^{\check{a}}$. This explains in a natural way the emergence of an extra complex structure when dealing with coisotropic branes [19, 27, 17]: imposing constraints linear in the fermionic derivatives does give rise to complex structures.

When analyzing the boundary term in the variation of the $N = 2$ action, eq. (4.36), one has to take into account that $\hat{X}$ is constrained by eq. (4.40). As a consequence we have that,

$$\delta X^{\check{a}} = \frac{\partial X^{\check{a}}}{\partial \tilde{X}^{\check{b}}} \left(\hat{D}\delta\Lambda^{\check{b}} - \tilde{K}^{\check{b}}{}_{\check{c}} D\delta\Lambda^{\check{c}} \right), \tag{4.41}$$

where $\tilde{X}$ are coordinates in which the complex structure K is constant (which we denote by $\tilde{K}$) and we expressed $\tilde{X}$ in terms of unconstrained fermionic superfields Λ: $\tilde{X}^{\check{a}} = \hat{D}\,\Lambda^{\check{a}} - \tilde{K}^{\check{a}}{}_{\check{b}}\, D\Lambda^{\check{b}}$. Using this in eq. (4.36), we find that it becomes,

$$\begin{aligned} \delta\mathcal{S}\Big|_{\text{boundary}} &= -i \int d\tau d^2\theta\, \delta\Lambda^{\check{e}} DX^{\check{b}} \frac{\partial X^{\check{a}}}{\partial \tilde{X}^{\check{e}}} \Big(2M_{\check{c}} K^{\check{c}}{}_{[\check{b},\check{a}]} + M_{\check{a},\check{c}} K^{\check{c}}{}_{\check{b}} - M_{\check{c},\check{b}} K^{\check{c}}{}_{\check{a}} \Big) \\ &\quad -i \int d\tau d^2\theta\, \delta\Lambda^{\check{e}} \frac{\partial X^{\check{a}}}{\partial \tilde{X}^{\check{e}}} \Big(M_{\check{a},\hat{b}} \hat{D}X^{\hat{b}} - M_{\check{c},\hat{b}} K^{\check{c}}{}_{\check{a}} DX^{\hat{b}} \Big), \end{aligned} \tag{4.42}$$

where,

$$M_{\check{a}} \equiv V_{\check{b}} J^{\check{b}}{}_{\check{a}} - W_{\check{a}}, \tag{4.43}$$

and where we denoted the coordinates $X^{\hat{\mu}}$ and $X^{\bar{\hat{\mu}}}$ collectively by $X^{\hat{a}}$. Using eq. (4.30) and the fact that $R^{\hat{\mu}}{}_{\bar{\check{\nu}}}$ does not depend on $\check{X}$, one shows that the second line in eq. (4.42) vanishes provided,

$$V_{\hat{\mu}\bar{\check{\nu}}} = V_{\check{\mu}\bar{\hat{\nu}}} = 0, \tag{4.44}$$

i.e. the Kähler potential factorizes (modulo a Kähler transformation) as $V = \hat{V}(\hat{X}, \bar{\hat{X}}) + \check{V}(\check{X}, \bar{\check{X}})$. We rewrite the argument of the first line in eq. (4.42) as,

$$\begin{aligned} &2M_{\check{c}} K^{\check{c}}{}_{[\check{b},\check{a}]} + M_{\check{a},\check{c}} K^{\check{c}}{}_{\check{b}} - M_{\check{c},\check{b}} K^{\check{c}}{}_{\check{a}} = \\ &\qquad 2F_{\check{a}\check{b}} + \partial_{\check{a}} \big(V_{\check{c}} (JK)^{\check{c}}{}_{\check{b}} - W_{\check{c}} K^{\check{c}}{}_{\check{b}} \big) - \partial_{\check{b}} \big(V_{\check{c}} (JK)^{\check{c}}{}_{\check{a}} - W_{\check{c}} K^{\check{c}}{}_{\check{a}} \big), \end{aligned} \tag{4.45}$$

where,

$$F_{\check{a}\check{b}} \equiv -\omega_{\check{a}\check{c}}\, K^{\check{c}}{}_{\check{b}} = -g_{\check{a}\check{c}} \left(JK \right)^{\check{c}}{}_{\check{b}}, \tag{4.46}$$

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with ω the Kähler form ($\omega_{ab} \equiv g_{ac} J^c{}_b$). From this we read that the boundary term in the variation, eq. (4.42), does vanish provided that F_{ab} is a closed 2-form. Locally we get that

$$F_{\check{a}\check{b}} = \partial_{\check{a}} A_{\check{b}} - \partial_{\check{b}} A_{\check{a}}, \tag{4.47}$$

with,

$$A_{\check{a}} = -\frac{1}{2} V_{\check{c}} (JK)^{\check{c}}{}_{\check{a}} + \frac{1}{2} W_{\check{c}} K^{\check{c}}{}_{\check{a}} + \partial_{\check{a}} f, \tag{4.48}$$

with f an arbitrary real function. Given F, eqs. (4.47) and (4.48) constrain the potential W. From the fact that $F_{\check{a}\check{b}}$ is antisymmetric in its indices we immediately find that both $F_{\check{a}\check{b}}$ and $\omega_{\check{a}\check{b}}$ are $(2,0) + (0,2)$ forms with respect to the complex structure K implying — as ω is non-degenerate — that $k = 2l$, $l \in \mathbb{N}$. As a consequence the dimension of the submanifold spanned by $X^{\check{a}}$ is a multiple of four [19, 27, 17]. So here we are dealing with open strings in the presence of a coisotropic D$(n+2l)$-brane. An obvious realization of the previous is given by the case in which the submanifold parametrized by the coordinates $X^{\check{a}}$ is hyper-Kähler.

The reason for calling these D-branes coisotropic is that they wrap coisotropic submanifolds. Denoting the submanifold wrapped by the D-brane by $\mathcal{N} \subset \mathcal{M}$, each tangent space of $\mathcal{N}$ is a subspace of the tangent space of $\mathcal{M}$, $T_{\mathcal{N}} \subset T_{\mathcal{M}}$. For $\mathcal{N}$ to be coisotropic, we need $T_{\mathcal{N}}^{\perp} \subset T_{\mathcal{N}}$, where $T_{\mathcal{N}}^{\perp}$ is the symplectic complement of $T_{\mathcal{N}}$.[14] The complement $T_{\mathcal{N}}^{\perp}$ is generated by those tangent vectors along the brane which are in the image of π_-. We will denote them by $\delta\sigma^{\hat{a}}$. These are symplectic orthogonal to themselves because of the relation eq. (4.30) and the symmetry of R. On the other hand they are symplectic orthogonal to all vectors in $\operatorname{Im}\pi_+$, because the factorization of the metric implies $\omega_{\hat{\mu}\bar{\nu}} = \omega_{\check{\mu}\bar{\nu}} = 0$. No other vectors of $\operatorname{Im}\pi_-$ can be orthogonal to the $\sigma^{\hat{a}}$, because ω is non- degenerate. This shows that indeed $T_{\mathcal{N}}^{\perp} = \{\delta\sigma^{\hat{a}}\} \subset T_{\mathcal{N}} = \{\delta\sigma^{\hat{a}}, \delta X^{\check{a}}\}$. Whenever $k = 0$, we find that $T_{\mathcal{N}}^{\perp} = T_{\mathcal{N}}$, so that, as mentioned before, $\mathcal{N}$ becomes lagrangian. In the other extreme, when $k = n$, we find a maximally coisotropic D(4l)-brane wrapping the entire target space $\mathcal{M}$. This is obviously only possible for target space dimensions which are a multiple of four. In general, the magnetic flux F, the pullback of ω and the additional complex structure $K = -\omega^{-1} F$ are only nonzero on the $4l$-dimensional quotient space $T_{\mathcal{N}}/T_{\mathcal{N}}^{\perp} = \{\delta X^{\check{a}}\}$, where they are all non-degenerate.

Upon using eq. (4.35) we can rewrite the non-standard boundary term in the $N = 1$ action, eq. (4.4), as,

$$\mathcal{S}_{\text{extra}} = i \int d\tau d\theta \left(V_{\check{a}} + W_{\check{b}} J^{\check{b}}{}_{\check{a}} \right) D' X^{\check{a}}. \tag{4.49}$$

Using the boundary condition eq. (4.40) and eq. (4.48) this indeed reduces to the standard boundary term eq. (2.9).

We will finish this section with an example for $n = 2$ (or $d = 4$). We have two twisted chiral fields which we denote by z and w. The Kähler potential is of the form $V(z - \bar{z}, w + \bar{w})$.

[14] Again, see appendix B for definitions.

Imposing Dirichlet boundary conditions,

$$\mathrm{Re}\, z = \text{ constant}, \qquad \mathrm{Im}\, w = \text{ constant}, \tag{4.50}$$

and Neumann boundary conditions,

$$D' \,\mathrm{Im}\, z = D' \,\mathrm{Re}\, w = 0, \tag{4.51}$$

we find that the action,

$$\mathcal{S} = \int d^2\sigma d^2\theta D' \hat{D}' \, V(z - \bar{z}, w + \bar{w}) \,, \tag{4.52}$$

describes open strings propagating on a Kähler manifold in the presence of a D2-brane wrapped around a lagrangian submanifold.

From the previous discussion we know that there exists the possibility of a (maximally) coisotropic brane — in the present case a D4-brane — as well. This can certainly be (locally) realized if the Kähler potential $V(z - \bar{z}, w + \bar{w})$ is actually hyper-Kähler, which is indeed so if the potential satisfies the Monge-Ampère equation,

$$V_{z\bar{z}} \, V_{w\bar{w}} - V_{z\bar{w}} \, V_{w\bar{z}} = 1. \tag{4.53}$$

The Legendre transform method [28] allows us to construct V in terms of a complex prepotential $h(x + z - \bar{z})$ with $x \in \mathbb{R}$. The Kähler potential is then given by the following Legendre transform,

$$V(z - \bar{z}, w + \bar{w}) = h(x + z - \bar{z}) + \bar{h}(x + \bar{z} - z) - x\,(w + \bar{w}). \tag{4.54}$$

Flat space corresponds e.g. to,

$$h = -\frac{1}{4}\,(x + z - \bar{z})^2 \,. \tag{4.55}$$

The metric can be expressed in terms of the prepotential,

$$\begin{aligned} g_{z\bar{z}} &= V_{z\bar{z}} = -4\,\frac{h''\bar{h}''}{h'' + \bar{h}''}\,, \\ g_{z\bar{w}} &= V_{z\bar{w}} = \frac{h'' - \bar{h}''}{h'' + \bar{h}''}\,, \\ g_{w\bar{z}} &= V_{w\bar{z}} = \frac{\bar{h}'' - h''}{h'' + \bar{h}''}\,, \\ g_{w\bar{w}} &= V_{w\bar{w}} = -\frac{1}{h'' + \bar{h}''}\,, \end{aligned} \tag{4.56}$$

where $h'' \equiv \partial_x^2 h(x + z - \bar{z})$ and similarly for $\bar{h}''$. The complex structure K is given by,

$$K^z{}_{\bar{z}} = g_{w\bar{z}}, \quad K^z{}_{\bar{w}} = g_{w\bar{w}}, \quad K^w{}_{\bar{z}} = -g_{z\bar{z}}, \quad K^w{}_{\bar{w}} = -g_{z\bar{w}}, \tag{4.57}$$

which, upon using eq. (4.46) and (4.56), gives the magnetic background,

$$F_{zw} = +i, \qquad F_{\bar{z}\bar{w}} = -i. \tag{4.58}$$

Using coordinates in which K is constant,

$$r \equiv z + \bar{z}, \quad s \equiv V_w, \quad t \equiv i(\bar{w} - w), \quad u \equiv iV_z, \tag{4.59}$$

one easily determines W such that the boundary term in the variation of the action eq. (4.5) vanishes,

$$W = \frac{i}{2}\left(z\, V_z + w\, V_w - \bar{z}\, V_{\bar{z}} - \bar{w}\, V_{\bar{w}}\right). \tag{4.60}$$

So the action eq. (4.1) with V given by eq. (4.54) and W by eq. (4.60) together with the Neumann boundary conditions,

$$\hat{D}z = V_{w\bar{z}}\, D\bar{z} + V_{w\bar{w}}\, D\bar{w}\,, \qquad \hat{D}w = -V_{z\bar{z}}\, D\bar{z} - V_{z\bar{w}}\, D\bar{w}\,, \tag{4.61}$$

describes open strings in the presence of a maximally coisotropic D4-brane. Taking flat space eq. (4.55), one recovers e.g. the example studied in [22].

5. Type B branes

We start from the most general $N = 2$ invariant action,

$$\mathcal{S} = -\int d^2\sigma d^2\theta D'\hat{D}'\, V(X, \bar{X}) + i \int d\tau d^2\theta\, W(X, \bar{X}), \tag{5.1}$$

where V and W are real scalar functions of the chiral superfields X and $\bar{X}$ which were defined in eq. (3.14). Working out the D' and $\hat{D}'$ derivatives we get,

$$\mathcal{S} = -2i \int d^2\sigma d^2\theta\, V_{\alpha\bar{\beta}}\left(DX^\alpha DX^{\bar{\beta}} - D'X^\alpha D'X^{\bar{\beta}}\right) + i \int d\tau d^2\theta\, W(X, \bar{X}). \tag{5.2}$$

Note that even in the presence of boundaries, the action remains invariant under Kähler transformations,

$$V(X, \bar{X}) \to V'(X, \bar{X}) = V(X, \bar{X}) + f(X) + \bar{f}(\bar{X}). \tag{5.3}$$

In addition we have the following invariance as well,

$$W(X, \bar{X}) \to W'(X, \bar{X}) = W(X, \bar{X}) + g(X) + \bar{g}(\bar{X}). \tag{5.4}$$

Performing the integral over $\hat{\theta}$ and comparing the result to the $N = 1$ action in eq. (2.7), we find that the target space is a Kähler manifold with Kähler potential $V(X, \bar{X})$ — i.e. the non-vanishing components of the metric are $g_{\alpha\bar{\beta}} = V_{\alpha\bar{\beta}}$ — which carries a U(1) bundle where the non-vanishing components of the magnetic field $F_{ab} \equiv b_{ab}$ are determined by the potential $W(X, \bar{X})$,

$$F_{\alpha\bar{\beta}} = -F_{\bar{\beta}\alpha} = -i\, W_{\alpha\bar{\beta}}, \qquad F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0. \tag{5.5}$$

The last equation states that we are dealing with a holomorphic vector bundle.

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When varying the action eqs. (5.1) or (5.2), one needs to take the constraints eqs. (3.14) or (3.15) into account. e.g. working in $N=(2,2)$ superspace, we express[15] X in terms of an unconstrained superfield L: $X^\alpha = \bar{\mathbb{D}}_+\bar{\mathbb{D}}_- L^\alpha = 2\,\bar{\mathbb{D}}'\,\bar{\mathbb{D}}\,L^\alpha$. In $N=2$ superspace one has unconstrained $N=2$ fields Λ^α, $\Lambda^{\bar\alpha}$ and M^α, $M^{\bar\alpha}$, in terms of which we get,

$$\begin{aligned} X^\alpha &= \left(\hat D - iD\right)\Lambda^\alpha, \qquad X^{\bar\alpha} = \left(\hat D + iD\right)\Lambda^{\bar\alpha}, \\ D'X^\alpha &= \left(\hat D - iD\right)M^\alpha - \partial_\sigma\Lambda^\alpha, \qquad D'X^{\bar\alpha} = \left(\hat D + iD\right)M^{\bar\alpha} + \partial_\sigma\Lambda^{\bar\alpha}. \end{aligned} \tag{5.6}$$

Using this we get the boundary term in the variation of the action eq. (5.1) or (5.2),

$$\begin{aligned} \delta\mathcal{S}\Big|_{\text{boundary}} = -2i\int d\tau d^2\theta\Big(&\delta\Lambda^\alpha\left(V_{\alpha\bar\beta}D'X^{\bar\beta} + iW_{\alpha\bar\beta}DX^{\bar\beta}\right) \\ &+\delta\Lambda^{\bar\alpha}\left(V_{\bar\alpha\beta}D'X^\beta - iW_{\bar\alpha\beta}DX^\beta\right)\Big). \end{aligned} \tag{5.7}$$

Once again we need suitable boundary conditions to cancel this. We impose Dirichlet boundary conditions on the unconstrained $N=2$ superfields Λ,

$$\delta\Lambda^\alpha = R^\alpha{}_\beta\,\delta\Lambda^\beta + R^\alpha{}_{\bar\beta}\,\delta\Lambda^{\bar\beta}. \tag{5.8}$$

As $(\hat D - iD)\Lambda^{\bar\alpha}$ should not appear in δX^α, we necessarily need that,

$$R^\alpha{}_{\bar\beta} = R^{\bar\alpha}{}_\beta = 0. \tag{5.9}$$

We find that $\delta X^\alpha = R^\alpha{}_\beta\,\delta X^\beta$ follows from $\delta\Lambda^\alpha = R^\alpha{}_\beta\,\delta\Lambda^\beta$ provided,

$$R^\alpha{}_{\delta,\bar\epsilon}\,\mathcal{P}^\delta_{+\beta}\,\mathcal{P}^{\bar\epsilon}_{+\bar\gamma} = 0, \tag{5.10}$$

is satisfied. Finally, requiring that $DX^\alpha = \mathcal{P}_+{}^\alpha{}_\beta\,DX^\beta$ and $\partial_\tau X^\alpha = \mathcal{P}^\alpha_{+\beta}\,\partial_\tau X^\beta$ are mutually compatible gives the condition,

$$R^\alpha{}_{\delta,\epsilon}\,\mathcal{P}_+{}^\delta{}_{[\beta}\,\mathcal{P}_+{}^\epsilon{}_{\gamma]} = 0. \tag{5.11}$$

Eqs. (5.10) and (5.11) guarantee the existence of coordinates $X^{\hat\alpha}$, $\hat\alpha \in \{k+1,\cdots m\}$ where k is the rank of $\mathcal{P}_+$, such that the Dirichlet boundary conditions are given by,

$$X^{\hat\alpha} = \text{ constant}. \tag{5.12}$$

Denoting the remainder of the coordinates by $X^{\tilde\alpha}$, $\tilde\alpha \in \{1,\cdots,k\}$, we find that eq. (5.7) vanishes provided we impose the Neumann boundary conditions,

$$V_{\tilde\alpha\bar\beta}\,D'X^{\bar\beta} = -i\,W_{\tilde\alpha\bar{\tilde\beta}}\,DX^{\bar{\tilde\beta}}, \tag{5.13}$$

where $\bar\beta$ runs from 1 through m.

In this situation the σ-model describes open strings in a background with a D2k-brane wrapped on a holomorphic submanifold. In addition, the D-brane can carry non-trivial magnetic flux as long as this corresponds to the curvature of a connection on a holomorphic line bundle. Note that, in contrast to the A brane case, the conditions on the U(1) flux are independent of the geometry of the brane.

[15] Once more, for conventions we refer to the appendix.

6. Duality transformations

6.1 Generalities

Supersymmetric non-linear σ-models allow for various duality transformations interchanging the different types of superfields [2, 29, 30, 9, 31, 32]. Here we are chiefly interested in duality transformations interchanging chiral and twisted chiral fields and vice-versa. Let us first briefly review the case without boundaries. The basic idea is to start with a potential with an isometry. Subsequently one gauges the isometry and imposes — using Lagrange multipliers — that the gauge fields are pure gauge. Integrating over the Lagrange multipliers gives back the original model while integrating over the gauge fields (or their potentials which are unconstrained superfields) yields the dual model.

We start from the $N = (2, 2)$ action (without boundaries),

$$\mathcal{S} = \int d^2\sigma\, d^4\theta \left(- \int dy\, W(y, \cdots) + (z + \bar{z})\, y \right), \tag{6.1}$$

where y is an unconstrained $N = (2, 2)$ superfield, z is either a chiral or a twisted chiral superfield and $\cdots$ stand for other, spectator fields. The equations of motion for y give,

$$z + \bar{z} = W(y, \cdots), \tag{6.2}$$

which upon inversion gives,

$$y = U(z + \bar{z}, \cdots). \tag{6.3}$$

Using this to eliminate y yields the second order action,

$$\mathcal{S} = \int d^2\sigma\, d^4\theta \int d(z + \bar{z})\, U(z + \bar{z}, \cdots). \tag{6.4}$$

When however taking z and $\bar{z}$ to be chiral and integrating over them in eq. (6.1) we get,

$$\bar{\mathbb{D}}_+ \bar{\mathbb{D}}_- y = \mathbb{D}_+ \mathbb{D}_- y = 0, \tag{6.5}$$

which is solved by putting $y = w + \bar{w}$ with w a twisted chiral superfield. If on the other hand we started off with a field z which was twisted chiral we get upon integrating over z and $\bar{z}$,

$$\bar{\mathbb{D}}_+ \mathbb{D}_- y = \mathbb{D}_+ \bar{\mathbb{D}}_- y = 0, \tag{6.6}$$

which is now solved by putting $y = w + \bar{w}$, with w a chiral superfield. The resulting second order action (which is the action one starts with) is in both cases given by,

$$\mathcal{S} = - \int d^2\sigma\, d^4\theta \int d(w + \bar{w})\, W(w + \bar{w}, \cdots). \tag{6.7}$$

So we conclude that this duality transformation — associated with a U(1) isometry — allows one to exchange chiral for twisted chiral fields and vice-versa. The natural question which arises here is whether this duality symmetry persists when boundaries are present. The main difficulty will be to introduce the right boundary terms such that the boundary conditions of the various fields remain consistent with the duality transformation.

6.2 From B to A branes

We start our investigation with B-branes which are fairly well under control. The initial model has n chiral fields z^α, $\alpha \in \{1, \cdots, n\}$ and it is characterized by a Kähler potential $V(z+\bar{z})$ and a U(1) prepotential $W(z+\bar{z})$. As the notation already indicates the potentials are such that $\partial_\alpha V = \partial_{\bar\alpha} V$ and $\partial_\alpha W = \partial_{\bar\alpha} W$ hold, implying the existence of n isometries which should allow us to dualize the model to an A-brane. The action is given by eq. (5.1) and we choose the boundary conditions as fully Neumann,

$$\begin{aligned} V_{\bar\alpha\beta}(z+\bar z)\, D' z^\beta &= +i W_{\bar\alpha\beta}(z+\bar z)\, D z^\beta, \\ V_{\alpha\bar\beta}(z+\bar z)\, D' z^{\bar\beta} &= -i W_{\alpha\bar\beta}(z+\bar z)\, D z^{\bar\beta}. \end{aligned} \tag{6.8}$$

We introduce a set of unconstrained real superfields $y^\alpha = (y^\alpha)^\dagger$ (the gauge fields which in a second order formulation of the model will be identified with $z^\alpha + z^{\bar\alpha}$) which satisfy the boundary conditions,

$$\begin{aligned} V_{\alpha\beta}(y)\, D' y^\beta &= +W_{\alpha\beta}(y)\, \hat D y^\beta, \\ V_{\alpha\beta}(y) \hat D' y^\beta &= -W_{\alpha\beta}(y)\, D y^\beta, \end{aligned} \tag{6.9}$$

where we used the isometries of V and W. The first order action is given by,

$$\begin{aligned} \mathcal{S} = &- \int d^2\sigma d^2\theta \Big(D'\hat D'\, V(y) - 2i\, w_\alpha\, \mathbb{D}_- \bar{\mathbb{D}}_+\, y^\alpha - 2i\, w_{\bar\alpha}\, \mathbb{D}_+ \bar{\mathbb{D}}_-\, y^\alpha \Big) \\ &\quad + i \int d\tau d^2\theta \left(W(y) - y^\alpha \frac{\partial W(y)}{\partial y^\alpha}\Big|_{y=y(w+\bar w)} - i\, y^\alpha \left(w_\alpha - w_{\bar\alpha}\right) \right) \\ = & \int d^2\sigma d^2\theta D'\hat D' \left(- V(y) + y^\alpha \left(w_\alpha + w_{\bar\alpha}\right)\right) \\ &\quad + i \int d\tau d^2\theta \left(W(y) - y^\alpha \frac{\partial W(y)}{\partial y^\alpha}\Big|_{y=y(w+\bar w)} \right), \end{aligned} \tag{6.10}$$

where the two forms of the action are related through partial integration and use of the constraints. When writing $y = y(w+\bar w)$, we mean that the y^α's are given as a function of the $w_\alpha + w_{\bar\alpha}$'s such that,

$$\frac{\partial V(y)}{\partial y^\alpha} = w_\alpha + w_{\bar\alpha}, \tag{6.11}$$

holds. In the first expression for the action, w_α and $w_{\bar\alpha}$ are unconstrained $N = 2$ superfields while in the second form for the action they are $N = (2, 2)$ twisted chiral superfields.

Varying w_α and $w_{\bar\alpha}$ in the first form of the action gives the bulk equation of motion,

$$\mathbb{D}_- \bar{\mathbb{D}}_+\, y^\alpha\Big|_{\theta'=\hat\theta'=0} = \mathbb{D}_+ \bar{\mathbb{D}}_-\, y^\alpha\Big|_{\theta'=\hat\theta'=0} = 0. \tag{6.12}$$

These constraints are themselves twisted chiral fields implying (by acting with D and $\hat D$ on them) that eq. (6.12) is equivalent to the full $N = (2, 2)$ superspace constraints,

$$\mathbb{D}_- \bar{\mathbb{D}}_+\, y^\alpha = \mathbb{D}_+ \bar{\mathbb{D}}_-\, y^\alpha = 0, \tag{6.13}$$

which are solved by putting,

$$y^\alpha = z^\alpha + z^{\bar\alpha}, \tag{6.14}$$

with z^α chiral superfields. The variation yields a boundary term as well which vanishes if we impose the Dirichlet boundary conditions on the Lagrange multipliers,

$$-i\big(w_\alpha - w_{\bar\alpha}\big) - \frac{\partial W(y)}{\partial y^\alpha}\Big|_{y=y(w+\bar w)} = \text{ constant.} \tag{6.15}$$

Going to the second order action and using eq. (6.9) we recover the original model describing open strings on a Kähler manifold with Kähler potential $V(z+\bar z)$ in the presence of a space-filling B-brane on which one has a holomorphic $\mathrm{U}(1)$ bundle determined by the prepotential $W(z+\bar z)$. The boundary conditions eq. (6.8) follow from combining eq. (6.9) with eq. (6.14).

We now turn to the dual model which one obtains by integrating the first order action (in the second form of eq. (6.10)) over the gauge fields y^α. Doing so, one finds eq. (6.11) as the bulk equations of motion. It implicitly gives the y^α's as a function of the twisted chiral superfields $w^\alpha + w^{\bar\alpha}$. Passing from the first order action eq. (6.10) to the second order action, we get the action for the dual model,

$$\mathcal{S} = \int d^2\sigma d^2\theta D'\hat D'\hat V(X,\bar X) + i\int d\tau d^2\theta\, \hat W(X,\bar X). \tag{6.16}$$

The resulting model is once more Kähler with the Kähler potential given by,

$$\hat V(w+\bar w) = -V(y(w+\bar w)) + (w_\alpha + w_{\bar\alpha})\, y^\alpha(w+\bar w). \tag{6.17}$$

The Kähler metric of the dual model is the inverse of the Kähler metric of the original model,

$$\frac{\partial^2\hat V}{\partial w_\alpha \partial w_{\bar\beta}} = \left(\frac{\partial^2 V}{\partial y^\alpha \partial y^\beta}\right)^{-1}\Big|_{y=y(w+\bar w)}. \tag{6.18}$$

The boundary potential is given by,

$$\begin{aligned}\hat W(w+\bar w) &= W(y(w+\bar w)) - y^\alpha\, \frac{\partial W(y)}{\partial y^\alpha}\Big|_{y=y(w+\bar w)}\\ &= W(y(w+\bar w)) - \frac{\partial\hat V}{\partial w_\alpha}\left(\frac{\partial^2\hat V}{\partial w_\alpha\partial w_{\bar\beta}}\right)^{-1}\frac{\partial W}{\partial w_{\bar\beta}}.\end{aligned} \tag{6.19}$$

The model has Dirichlet boundary conditions given by eq. (6.15) which can be rewritten as,

$$-i\big(w_\alpha - w_{\bar\alpha}\big) - \left(\frac{\partial^2\hat V}{\partial w_\alpha\partial w_{\bar\beta}}\right)^{-1}\frac{\partial W(y(w+\bar w))}{\partial w_{\bar\beta}} = \text{ constant,} \tag{6.20}$$

and a set of Neumann boundary conditions which either follow from eq. (6.15) using the constraints eq. (3.17) or which can be obtained by acting with D' and $\hat D$ on eq. (6.11) and using eq. (6.9). One verifies that the boundary term in the variation of the action (see eq. (4.5)) indeed vanishes.

6.3 From A to B branes

6.3.1 Dualizing lagrangian branes

We start from the D1-brane discussed in section 4, assuming the existence of an isometry. The σ-model is parametrized by a single twisted chiral field w (and its complex conjugate $\bar{w}$) with Kähler potential $V(w+\bar{w})$ and boundary potential $W(w+\bar{w})$. So we have $V_w = V_{\bar{w}}$ and $W_w = W_{\bar{w}}$. The action is given in eq. (4.1) and the Dirichlet boundary condition is

$$(V_w + i\,W_w)\,\delta w = (V_{\bar{w}} - i\,W_{\bar{w}})\,\delta \bar{w}. \tag{6.21}$$

The Neuman boundary condition which follows from this is,

$$(V_w + i\,W_w)\,D' w + (V_{\bar{w}} - i\,W_{\bar{w}})\,D'\bar{w} = 0. \tag{6.22}$$

We introduce a real gauge (unconstrained) superfield y satisfying the boundary condition,

$$\mathbb{D}' y = -i\,\frac{W_y(y)}{V_y(y)}\,\mathbb{D}y, \qquad \bar{\mathbb{D}}' y = +i\,\frac{W_y(y)}{V_y(y)}\,\bar{\mathbb{D}}y. \tag{6.23}$$

The first order action is given by,

$$\begin{aligned}\mathcal{S} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}' \left\{V(y) - i\,u\,\bar{\mathbb{D}}\bar{\mathbb{D}}' y - i\,\bar{u}\,\mathbb{D}\mathbb{D}' y\right\} + \\ &i\int d\tau\, d^2\theta \left\{W(y) + \bar{\mathbb{D}}' u\left(\bar{\mathbb{D}}' y - i\,\frac{W_y(y)}{V_y(y)}\,\bar{\mathbb{D}}y\right) - \mathbb{D}'\bar{u}\left(\mathbb{D}' y + i\,\frac{W_y(y)}{V_y(y)}\,\mathbb{D}y\right)\right\},\end{aligned} \tag{6.24}$$

where the Lagrange multipliers u and $\bar{u} = u^\dagger$ are unconstrained complex $N = (2,2)$ superfields. Integrating over the Lagrange multipliers yields a bulk term,

$$\bar{\mathbb{D}}\bar{\mathbb{D}}'\, y = \mathbb{D}\mathbb{D}' y = 0, \tag{6.25}$$

which is solved in terms of a twisted chiral superfield w,

$$y = w + \bar{w}. \tag{6.26}$$

From the last term in eq. (6.24) we get a boundary condition as well which is equal to the one in eq. (6.23). Combining the boundary condition with eq. (6.26) and the bulk constraints,

$$\begin{aligned}\mathbb{D}w &= +\mathbb{D}' w, \quad \bar{\mathbb{D}}w = -\bar{\mathbb{D}}' w, \\ \mathbb{D}\bar{w} &= -\mathbb{D}'\bar{w}, \quad \bar{\mathbb{D}}\bar{w} = +\bar{\mathbb{D}}'\bar{w},\end{aligned} \tag{6.27}$$

which are equivalent to eq. (3.16), gives the original boundary conditions eqs. (6.21) and (6.22). Going to the second order action one recovers the original model.

We introduce a potential $Q(y)$ implicitely defined by,

$$W(y) = Q(y) - \frac{V'(y)Q'(y)}{V''(y)}\,, \tag{6.28}$$

where primes denote derivatives with respect to y. Using this and partial integration,[16] we can rewrite eq. (6.24) as,

$$\mathcal{S} = \int d^2\sigma\, d^2\theta\, D'\hat{D}' \Big\{ V(y) - y\,(z+\bar{z}) \Big\} + i \int d\tau\, d^2\theta \left\{ Q(y) - \frac{V'(y)Q'(y)}{V''(y)} + \frac{Q'(y)}{V''(y)}\,(z+\bar{z}) \right\}, \tag{6.29}$$

where we introduced the chiral superfield z,

$$z \equiv i\,\bar{\mathbb{D}}\bar{\mathbb{D}}'\, u, \qquad \bar{z} \equiv i\,\mathbb{D}\mathbb{D}'\,\bar{u}, \tag{6.30}$$

which by construction satisfy the constraints,

$$\bar{\mathbb{D}}z = \bar{\mathbb{D}}'z = \mathbb{D}\bar{z} = \mathbb{D}'\bar{z} = 0. \tag{6.31}$$

Integrating over the unconstrained superfield y gives the bulk equation of motion,

$$z + \bar{z} = \frac{\partial\, V(y)}{\partial\, y}\,, \tag{6.32}$$

which upon inversion gives y as a function of $z+\bar{z}$: $y = y(z+\bar{z})$. The boundary term arising from varying y,

$$\delta\mathcal{S}\Big|_{\text{boundary}} = -i \int d\tau\, d^2\theta\, \delta y \left(\frac{Q'(y)}{V''(y)} \right)' \big(V'(y) - (z+\bar{z}) \big), \tag{6.33}$$

vanishes by virtue of eq. (6.32). Using eqs. (6.32) and (6.31), we get from eq. (6.23) the Neumann boundary conditions,

$$\begin{aligned} \mathbb{D}'z &= -i\, \frac{W'\big(y(z+\bar{z})\big)}{V'\big(y(z+\bar{z})\big)}\, \mathbb{D}z, \\ \bar{\mathbb{D}}'\bar{z} &= +i\, \frac{W'\big(y(z+\bar{z})\big)}{V'\big(y(z+\bar{z})\big)}\, \bar{\mathbb{D}}\bar{z}. \end{aligned} \tag{6.34}$$

We now go to the second order action. In order to make this as explicit as possible, we introduce a potential $P(y)$ defined by

$$V(y) = -\int dy\, P(y). \tag{6.35}$$

With this eq. (6.32) can be rewritten as,

$$z + \bar{z} = P(y), \tag{6.36}$$

or,

$$y = P^{-1}(z+\bar{z}). \tag{6.37}$$

[16]The calculations are facilitated by using $\int d^2\sigma\, d^2\theta\, D'\hat{D}' = -(1/4)\int d^2\sigma\, \mathbb{D}\bar{\mathbb{D}}\mathbb{D}'\bar{\mathbb{D}}'$ and $\int d\tau\, d^2\theta = -(i/2)\int d\tau\, \mathbb{D}\bar{\mathbb{D}}$. Once again we refer to appendix A for conventions.

Using this, the second order action follows from eq. (6.29):

$$\mathcal{S} = -\int d^2\sigma\, d^2\theta\, D'\hat{D}' \int d\big(z+\bar{z}\big)\, P^{-1}(z+\bar{z}) + i\int d\tau\, d^2\theta\, Q\big(P^{-1}(z+\bar{z})\big), \quad (6.38)$$

from which we read the Kähler potential $\hat{V}(z+\bar{z})$ and the $\mathrm{U}(1)$ potential $\hat{W}(z+\bar{z})$:

$$\hat{V}(z+\bar{z}) = \int d(z+\bar{z})\, P^{-1}(z+\bar{z}), \qquad \hat{W}(z+\bar{z}) = Q\big(P^{-1}(z+\bar{z})\big). \quad (6.39)$$

In terms of the dual variables we can rewrite the boundary conditions eq. (6.34) as,

$$D'z = +i\,\frac{\hat{W}_{z\bar{z}}}{\hat{V}_{z\bar{z}}}\, Dz, \qquad D'\bar{z} = -i\,\frac{\hat{W}_{z\bar{z}}}{\hat{V}_{z\bar{z}}}\, D\bar{z}, \quad (6.40)$$

which are recognized as the standard Neumann boundary conditions in the presence of magnetic background field.

Concluding we find that the dual theory describes open strings in the presence of a space filling D2 B-brane on a Kähler manifold whose potential is given by $\hat{V} = \int d(z+\bar{z})\, P^{-1}(z+\bar{z})$. In addition, a $\mathrm{U}(1)$ bundle with potential $\hat{W} = Q\big(P^{-1}(z+\bar{z})\big)$ is present as well.

6.3.2 Dualizing coisotropic branes

Now let us look at the case of coisotropic A branes. The example discussed at the end of section 4 is characterized by a Kähler potential $V(z-\bar{z}, w+\bar{w})$ which satisfies the Monge-Ampère equation eq. (4.53). The potential has an obvious isometry,

$$\delta z = -\varepsilon_1, \qquad \delta w = -i\,\varepsilon_2, \quad (6.41)$$

with $\varepsilon_1,\ \varepsilon_2 \in \mathbb{R}$ and constant. However the boundary potential W given in eq. (4.60) can be rewritten as,

$$W = \frac{i}{2}\Big(\big(z+\bar{z}\big)\, V_z(z-\bar{z}, w+\bar{w}) + \big(w-\bar{w}\big)\, V_w(z-\bar{z}, w+\bar{w})\Big), \quad (6.42)$$

and does not exhibit the above mentioned isometry. Remarkably one finds — using the fact that the Kähler potential satisfies the Monge-Ampère equation eq. (4.53) — that $\hat{D}\,W$ transforms in a total D derivative, making the boundary term in the action invariant as well. Let us make this very explicit by making a change of coordinates:

$$\begin{aligned} z_1 &\equiv z+\bar{z} - i\,V_w, \qquad \bar{z}_1 \equiv z+\bar{z}+iV_w, \\ z_2 &\equiv -i\big(w-\bar{w}\big) + V_z, \qquad \bar{z}_2 \equiv -i\big(w-\bar{w}\big) - V_z. \end{aligned} \quad (6.43)$$

One verifies that both z_1 and z_2 are chiral boundary fields, i.e.,

$$\hat{D}z_a = +iDz_a, \qquad \hat{D}\bar{z}_a = -iD\bar{z}_a \quad (6.44)$$

for $a \in \{1,2\}$. The boundary potential W given in eq. (6.42) can be rewritten as,

$$W = \frac{i}{8}\Big(\big(z_1+\bar{z}_1\big)\big(z_2-\bar{z}_2\big) - \big(z_1-\bar{z}_1\big)\big(z_2+\bar{z}_2\big)\Big). \quad (6.45)$$

The isometry eq. (6.41) becomes in these coordinates

$$\delta z_1 = \delta \bar{z}_1 = -2\,\varepsilon_1, \qquad \delta z_2 = \delta \bar{z}_2 = -2\,\varepsilon_2. \tag{6.46}$$

Under these transformations, the potential transforms as

$$\delta W = -\frac{i}{2}\Big(\big(\varepsilon_1 z_2 - \varepsilon_2 z_1\big) - \big(\varepsilon_1 \bar{z}_2 - \varepsilon_2 \bar{z}_1\big)\Big), \tag{6.47}$$

which — by virtue of the constraints eq. (6.44) — gives $\delta \int d\tau d^2\theta\, W = 0$. The present situation is similar to the one studied in [33]. In order to gauge the isometries, one needs first to modify the potential W such that it becomes invariant under the isometries. This is achieved by modifying W to W',

$$W' = W + \frac{i}{2}\Big(q\, z_1 z_2 - q\, \bar{z}_1 \bar{z}_2 + \xi - \bar{\xi}\Big), \tag{6.48}$$

where $q \in \mathbb{R}$ and ξ is a new (auxiliary) boundary *chiral* field which transforms under the isometry as,

$$\delta \xi = (1 + 2q)\, \varepsilon_1 z_2 - (1 - 2q)\, \varepsilon_2 z_1. \tag{6.49}$$

With this one gets that $\delta W' = 0$. Because the difference between W and W' is the sum of a holomorphic and an anti-holomorphic function of the boundary chiral fields we have that $\int d\tau d^2\theta\, W' = \int d\tau d^2\theta\, W$, so the physical content of the model remains unchanged. However — as was shown in [33] — when trying to gauge more than one isometry simultanously one can encounter an obstruction (which was given a Lie algebra cohomology interpretation in [34]) which renders gauging of the full isometry algebra impossible. In the present situation this obstruction is indeed present — as one can check using the equations developed in [33] — implying that we can only gauge a linear combination of the isometries given in eqs. (6.46) and (6.49).

For concreteness, we will gauge the ε_2 isometry. Our analysis is considerably simplified if we rewrite the boundary term in the action as,

$$\begin{aligned} \mathcal{S}\Big|_{\text{boundary}} &= i \int d\tau d^2\theta\, W = i \int d\tau d^2\theta\, \frac{i}{4}\big(z_1 + \bar{z}_1\big)\big(z_2 - \bar{z}_2\big) \\ &= i \int d\tau d^2\theta\, i\,\big(z + \bar{z}\big)\, V_z. \end{aligned} \tag{6.50}$$

The gauging procedure is now simple. We introduce an unconstrained gauge field y satisfying the boundary conditions,

$$\mathbb{D}' y = +i\, \mathbb{D} V_z(z - \bar{z}, y), \qquad \bar{\mathbb{D}}' y = +i\, \bar{\mathbb{D}} V_z(z - \bar{z}, y), \tag{6.51}$$

$$\mathbb{D}'\big(z - \bar{z}\big) = -i\, \mathbb{D} V_y(z - \bar{z}, y), \qquad \bar{\mathbb{D}}'\big(z - \bar{z}\big) = -i\, \bar{\mathbb{D}} V_y(z - \bar{z}, y). \tag{6.52}$$

The first order action is given by,

$$\begin{aligned} \mathcal{S} = &\int d^2\sigma\, d^2\theta\, D' \hat{D}' \Big\{ V(z - \bar{z}, y) - i\, u\, \bar{\mathbb{D}} \bar{\mathbb{D}}' y - i\, \bar{u}\, \mathbb{D} \mathbb{D}' y \Big\} \\ &+ i \int d\tau\, d^2\theta \Big\{ i(z + \bar{z}) V_z(z - \bar{z}, y) + \bar{\mathbb{D}}' u \left(\bar{\mathbb{D}}' y - i\, \bar{\mathbb{D}} V_z(z - \bar{z}, y) \right) \\ &- \mathbb{D}' \bar{u} \left(\mathbb{D}' y - i\, \mathbb{D} V_z(z - \bar{z}, y) \right) \Big\}, \end{aligned} \tag{6.53}$$

where u and $\bar{u} \equiv u^\dagger$ are unconstrained complex $N = (2,2)$ superfields. Integrating over u and $\bar{u}$ gives the equation of motion,

$$\mathbb{D}\mathbb{D}' y = \bar{\mathbb{D}}\bar{\mathbb{D}}' y = 0, \tag{6.54}$$

which is solved by putting,

$$y = w + \bar{w}, \tag{6.55}$$

with w a twisted chiral superfield. Varying y yields a boundary term as well which vanishes if we impose eq. (6.51). So the action eq. (6.53) together with the boundary condition eq. (6.52) reproduces upon integrating over u and $\bar{u}$ the original theory.

Partially integrating, we rewrite eq. (6.53) as,

$$\begin{aligned}\mathcal{S} = &\int d^2\sigma\, d^2\theta\, D'\hat{D}' \Big\{V(z-\bar{z},y) - y\left(r+\bar{r}\right)\Big\}\\ &+i\int d\tau\, d^2\theta \Big\{i\big(z+\bar{z}\big) - \big(r-\bar{r}\big)\Big\}\, V_z(z-\bar{z},y),\end{aligned} \tag{6.56}$$

where we introduced the chiral field r (and $\bar{r}$),

$$r \equiv i\,\bar{\mathbb{D}}\bar{\mathbb{D}}' u, \qquad \bar{r} \equiv i\,\mathbb{D}\mathbb{D}'\bar{u}. \tag{6.57}$$

Integrating over y yields the equation of motion,

$$V_y(z-\bar{z},y) = r + \bar{r}. \tag{6.58}$$

In terms of the prepotential h introduced in eq. (4.54), we can write a second order expression for the integrand of the bulkterm in eq. (6.56) as,

$$\Big(V(z-\bar{z},y) - y\left(r+\bar{r}\right)\Big)\Big|_{y=y(z-\bar{z},r+\bar{r})} = h(z-\bar{z}-r-\bar{r}) + \bar{h}(\bar{z}-z-r-\bar{r})\,. \tag{6.59}$$

Furthermore, requiring that the boundary term in the variation of y vanishes gives the Dirichlet boundary condition,

$$\operatorname{Im} r - \operatorname{Re} z = 0. \tag{6.60}$$

Combining eqs. (6.52) and (6.58) yields a Dirichlet,

$$\operatorname{Im} r - \operatorname{Re} z = \text{constant}, \tag{6.61}$$

and a Neumann,

$$-i\left(D'z - D'\bar{z}\right) = -D\big(r+\bar{r}\big), \tag{6.62}$$

boundary condition. Note that eqs. (6.60) and (6.61) are mutually compatible if we choose the constant in eq. (6.61) to be zero. Finally, the combination of eq. (6.51) and (6.58) yields two more Neumann boundary conditions,

$$\begin{aligned}-i\left(D'r - D'\bar{r}\right) + D'z + D'\bar{z} &= 0,\\ D'r + D'\bar{r} &= -i\,D\big(z-\bar{z}\big).\end{aligned} \tag{6.63}$$

So this implies that the open strings are propagating in a background which contains a D3-brane whose location is fixed by eq. (6.60). The bulk geometry is bi-hermitean and parametrized by a chiral, r, and a twisted chiral field, z, with the generalized Kähler potential given by $h(z-\bar{z}-r-\bar{r})+\bar{h}(\bar{z}-z-r-\bar{r})$. The non-vanishing components of the metric and the Kalb-Ramond form can be obtained from eq. (3.9) and are given by,

$$\begin{aligned} g_{r\bar{r}} = g_{z\bar{z}} &= +h''(z-\bar{z}-r-\bar{r})+\bar{h}''(\bar{z}-z-r-\bar{r}), \\ b_{r\bar{z}} = g_{z\bar{r}} &= -h''(z-\bar{z}-r-\bar{r})+\bar{h}''(\bar{z}-z-r-\bar{r}). \end{aligned} \tag{6.64}$$

Models whose bulk geometry is generalized Kähler will be studied in detail in [23]. Nonetheless, the previous example clearly shows that coisotropic branes do appear as duals to other brane configurations.

7. Conclusions and discussion

In this paper we presented a completely local $N=2$ superspace formulation of two-dimensional nonlinear σ-models for target spaces parameterized exclusively by chiral or twisted chiral fields (meaning that the bulk geometry is Kähler). This was possible because, contrary to previous attempts, only the supersymmetries which are preserved by the boundary conditions were required to remain manifest at all times. Starting from this formalism, a general $N=2$ superspace description of both A and B branes on Kähler manifolds was given. Interchanging type A boundary conditions for type B and vice versa turns out to be equivalent to exchanging chiral for twisted chiral superfields and vice versa allowing us without loss of generality to limit ourselves to type B boundary conditions. In this setting A-branes (B-branes) are described solely in terms of twisted-chiral (chiral) superfields.

The $N=2$ superspace description of type A branes turned out to be subtle. It gives rise to a "non-standard" boundary coupling which was shown to reduce to the standard one when proper use is made of the nontrivial boundary conditions. An open question – for the case of A-branes — is to find a better characterization or geometric interpretation of the boundary potential W. Perhaps a reformulation of the problem in terms of generalized complex geometry might shed some light.

The duality transformations relating A and B models were investigated as well. The main difficulty here is the identification of the right boundary terms in the first order action which see to it that boundary conditions correctly carry over during the duality transformation. When isometries are present, it is reasonably straightforward to dualize lagrangian A-branes to space filling B-branes and vice-versa. Dualizing a coisotropic A-brane turns out to be subtle. The example of a space-filling D4 coisotropic brane was shown to have two isometries. However only a linear combination of those two can be gauged. As a consequence we can dualize the model to a D3-brane where the bulk is now not Kähler anymore, but exhibits a bihermitean geometry.

It is clear that in general not sufficient isometries will be present to convert an A brane on a Kähler manifold to a B-brane on a Kähler manifold or vice-versa. When only part of the chiral or twisted chiral superfields can be dualized, the dual model will exhibit a bihermitian — or equivalently, a generalized Kähler — geometry. The study of these duality transformations will be reported on in [23].

Finally, $N = 2$ superspace provides a powerful framework for investigating the quantum properties of these non-linear σ-models (as was demonstrated in e.g. [35]). Requiring the β-functions to vanish gives rise to further conditions on the background geometry. E.g. at one loop one gets that the holomorphic bundle for a type B brane needs to satisfy a deformed stability condition as well. In this context it would be most interesting to calculate the one loop β-function for a coisotropic brane and make contact with the stability conditions obtained in [36].

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A. Conventions, notations and identities

We denote the worldsheet coordinates by $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}$, $\sigma \geq 0$. Sometimes we use worldsheet light-cone coordinates,

$$\sigma^{\ddagger} = \tau + \sigma, \qquad \sigma^{=} = \tau - \sigma. \tag{A.1}$$

The $N = (1,1)$ (real) fermionic coordinates are denoted by θ^+ and θ^- and the corresponding derivatives satisfy,

$$D_+^2 = -\frac{i}{2}\,\partial_{\ddagger}, \qquad D_-^2 = -\frac{i}{2}\,\partial_{=}\,, \qquad \{D_+, D_-\} = 0. \tag{A.2}$$

Passing from $N = (1,1)$ to $N = (2,2)$ superspace requires the introduction of two more real fermionic coordinates $\hat{\theta}^+$ and $\hat{\theta}^-$ where the corresponding fermionic derivatives satisfy,

$$\hat{D}_+^2 = -\frac{i}{2}\,\partial_{\ddagger}, \qquad \hat{D}_-^2 = -\frac{i}{2}\,\partial_{=}\,, \tag{A.3}$$

and again all other — except for (A.2) — (anti-)commutators do vanish. Quite often a complex basis is used,

$$\mathbb{D}_\pm \equiv \hat{D}_\pm + i\,D_\pm, \qquad \bar{\mathbb{D}}_\pm \equiv \hat{D}_\pm - i\,D_\pm, \tag{A.4}$$

which satisfy,

$$\{\mathbb{D}_+, \bar{\mathbb{D}}_+\} = -2i\,\partial_{\ddagger}, \qquad \{\mathbb{D}_-, \bar{\mathbb{D}}_-\} = -2i\,\partial_{=}, \tag{A.5}$$

and all other anti-commutators do vanish.

When dealing with boundaries in $N = (2,2)$ superspace, we introduce various derivatives as linear combinations of the previous ones. We summarize their definitions together with the non-vanishing anti-commutation relations. We have,

$$\begin{aligned} D &\equiv D_+ + D_-, \qquad \hat{D} \equiv \hat{D}_+ + \hat{D}_-, \\ D' &\equiv D_+ - D_-, \qquad \hat{D}' \equiv \hat{D}_+ - \hat{D}_-, \end{aligned} \tag{A.6}$$

with,

$$\begin{aligned} D^2 = \hat{D}^2 = D'^2 = \hat{D}'^2 &= -\frac{i}{2}\partial_\tau, \\ \{D, D'\} = \{\hat{D}, \hat{D}'\} &= -i\partial_\sigma. \end{aligned} \tag{A.7}$$

In addition we also use,

$$\begin{aligned} \mathbb{D} &\equiv \mathbb{D}_+ + \mathbb{D}_- = \hat{D} + i\,D, \qquad \mathbb{D}' \equiv \mathbb{D}_+ - \mathbb{D}_- = \hat{D}' + i\,D', \\ \bar{\mathbb{D}} &\equiv \bar{\mathbb{D}}_+ + \bar{\mathbb{D}}_- = \hat{D} - i\,D, \qquad \bar{\mathbb{D}}' \equiv \bar{\mathbb{D}}_+ - \bar{\mathbb{D}}_- = \hat{D}' - i\,D'. \end{aligned} \tag{A.8}$$

They satisfy,

$$\begin{aligned} \{\mathbb{D}, \bar{\mathbb{D}}\} = \{\mathbb{D}', \bar{\mathbb{D}}'\} &= -2i\,\partial_\tau, \\ \{\mathbb{D}, \bar{\mathbb{D}}'\} = \{\mathbb{D}', \bar{\mathbb{D}}\} &= -2i\,\partial_\sigma\,. \end{aligned} \tag{A.9}$$

B. Submanifolds of symplectic manifolds

A symplectic manifold $\mathcal{M}$ is a manifold endowed with a non-degenerate closed two-form ω. There are several natural ways to define specific submanifolds of these. We will do this by first defining the symplectic complement of a subspace of a symplectic vector space.

So let V be a symplectic vector space of dimension $d = 2n$. This means that it is equipped with a non-degenerate, skew-symmetric, bilinear form ω, called the symplectic form. The symplectic complement of a subspace W is defined as,

$$W^\perp = \{v \in V | \omega(v, w) = 0, \forall w \in W\}. \tag{B.1}$$

This satisfies $(W^\perp)^\perp = W$ and $\dim W + \dim W^\perp = \dim V$. However, contrary to the orthogonal complement (defined with a metric), generically $W \cap W^\perp \neq \emptyset$.

We are interested in the three following cases,

Isotropic: When $W \subseteq W^\perp$, W is called isotropic. This is true if and only if ω restricts to zero on W. Every one-dimensional subspace is isotropic.

Coisotropic: When $W^\perp \subseteq W$, W is called coisotropic. In other words, W is coisotropic if and only if $W^\perp$ is isotropic. Equivalently, W is coisotropic if and only if ω descends to a non-degenerate form on the quotient space $W/W^\perp$. A codimension one subspace is always coisotropic.

Lagrangian: When $W = W^\perp$, W is called Lagrangian, so that a Lagrangian subspace is both isotropic and coisotropic.

These definitions immediately imply that, because of the non-degeneracy of ω, a Lagrangian subspace is n-dimensional, where $n = d/2$. The number of dimensions of an isotropic (a coisotropic) subspace in necessarily smaller (bigger) than n.

Given a symplectic manifold $\mathcal{M}$, a submanifold $\mathcal{N}$ is called isotropic, coisotropic or Lagrangian if the tangent space $T_\mathcal{N}$ is an isotropic, coisotropic or Lagrangian subspace of $T_\mathcal{M}$, that is, if $T_\mathcal{N} \subseteq T_\mathcal{N}^\perp$, $T_\mathcal{N}^\perp \subseteq T_\mathcal{N}$ or $T_\mathcal{N} = T_\mathcal{N}^\perp$, respectively.

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